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STABILITY AND VIBRATIONS OF THICK-WALLED TUBES SUBJECTED TO FINITE TWIST AND EXTERNAL PRESSURE

A. ERTEPINAR†
Department of Engineering Sciences, Middle East Technical University, Ankara, Turkey

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Abstract—Stability and small vibrations of long, thick-walled, circular cylindrical tubes subjected to finite twist and external pressure are investigated using the theory of finite elastic deformations in conjunction with the theory of small deformations superposed on large elastic deformations. The material of the tube is assumed to be isotropic, elastic, homogeneous and incompressible. A numerical scheme is adopted to solve the system of partial differential equations and the associated boundary conditions governing the problem. The effect of finite twist on the frequencies and the loss of stability due to uniform external pressure is displayed by various curves relating the frequencies to initial radial deformation parameter.

INTRODUCTION

The theory of small deformations superposed on large elastic deformations [1] is now one of the most commonly adopted methods in the analysis of stability of thick and thick-walled bodies subjected to large elastic deformations. References [2–5] are among the notable works in this field published between 1955 and 1969. The related dynamic problems have been treated by Refs. [6–12]. With the exception of [3] and a recent paper by Patterson [13], the finite deformations were assumed to be caused either by a uniform external pressure, or a finite twist, or an end thrust.

The present work is a simple but interesting extension of the studies reported in [10] and [11]. Namely, the effect of finite twist on the stability and small vibrations of long, thick-walled, circular cylindrical tubes under external pressure is investigated. The material of the shell is assumed to be homogeneous, isotropic, perfectly elastic, and incompressible. The tube is first subjected to a finite twist and an external pressure and is then exposed to secondary dynamical displacements. The governing equations of the finitely deformed state are obtained by using the theory of finite elastic deformations [14, 15] while those governing the secondary displacement field are obtained through the use of the theory of small displacements superposed on large elastic deformations [1]. The field equations specialized for tubes made of a neo-Hookean material are solved numerically by the method of complementary functions [16, 17] in conjunction with the Runge–Kutta method of integration. Except for the case when both the axial and the circumferential wave numbers are zero, the oscillatory characteristics of the tubes are found to depend both on the amount of finite twist and the external pressure. When the frequency of oscillations ceases to be real-valued, the tube becomes unstable.

FORMULATION OF THE PROBLEM

1. *Finite deformation state*

A long, circular cylindrical tube of arbitrary wall thickness and made of a neo-Hookean material with a strain energy density function $W(I)$, where I is the first strain invariant, is subjected to a finite twist and a uniform external pressure. The inner and the outer radii of the undeformed tube are respectively denoted by A_1 and A_2. A state of finite deformation is assumed such that a material point at coordinates (r, θ, z) in the deformed tube is originally at coordinates $(R, \theta - \alpha z, z)$ where α is the angle of twist per unit length and R is a function of r only. The corresponding stress components are given by (see, e.g. Ref. [15]):

$$\tau^{11}(r) = \Phi\left\{\ln\frac{[(1-\bar{K})(A_1^2\bar{K}+r^2)]^{1/2}}{r} + \frac{1}{2}\left(\frac{A_1^2\bar{K}+r^2}{r^2} - \frac{1}{1-\bar{K}}\right) - \frac{\alpha^2 A_1^2}{2}\left(1-\bar{K}-\frac{r^2}{A_1^2}\right)\right\} - q_1, \tag{1}$$

†Presently at: Department of Mechanical Engineering and Mechanics, Drexel University, Philadelphia, PA 19104, U.S.A., on leave of absence from METU for the academic year of 1976–77.

$$\tau^{22}(r) = \frac{1}{r^2}\left[\tau^{11}(r) + \Phi\left(\frac{r^2}{A_1^2\bar{K}+r^2} - \frac{A_1^2\bar{K}+r^2}{r^2} + \alpha^2 r^2\right)\right], \tag{2}$$

$$\tau^{33}(r) = \tau^{11}(r) - \Phi\frac{A_1^2\bar{K}}{r^2}, \tag{3}$$

$$\tau^{23} = \Phi\alpha, \tag{4}$$

$$\tau^{12} = \tau^{31} = 0, \tag{5}$$

where $\Phi = 2(\partial W/\partial I)$, a_1 = inner radius of the deformed tube, q_1 = uniform internal pressure, and

$$\bar{K} = 1 - (a_1/A_1)^2. \tag{6}$$

Here, it is noted that, due to material incompressibility, $R^2 - r^2 = A_1^2 - a_1^2$, and $\bar{K}$ is also equal to $(R^2 - r^2)/A_1^2$. Hence, eqns (2) and (3) coincide with eqns (3.4.16) on p. 89 of Ref. [15] when specialized to the deformations considered in this work.

The external forces required to produce the prescribed deformation field are the resultant end torque

$$M = \frac{\pi}{2}\Phi\alpha A_1^4\left[\frac{(1-\lambda^2\bar{K})^2}{\lambda^4} - (1-\bar{K})^2\right], \tag{7}$$

the resultant normal force at the ends of the tube

$$N = 2\pi\Phi\int_{a_1}^{a_2}\left\{\left[r\ln\frac{[(1-\bar{K})(A_1^2\bar{K}+r^2)]^{1/2}}{r} + \frac{r}{2}\left(\frac{r^2 - A_1^2\bar{K}}{r^2} - \frac{1}{1-\bar{K}}\right) - \frac{\alpha^2 A_1^2 r}{2}\left(1 - \bar{K} - \frac{r^2}{A_1^2}\right)\right] - q_1\right\}\mathrm{d}r \tag{8}$$

and uniform external and internal pressures q_1, q_2 given by

$$q = q_2 - q_1 = \frac{1}{2}\Phi\left[\frac{\bar{K}(1-\lambda^2)}{(1-\bar{K})(1-\lambda^2\bar{K})} - \ln\frac{(1-\bar{K})}{(1-\lambda^2\bar{K})} - \frac{\alpha^2 A_1^2(1-\lambda^2)}{\lambda^2}\right] \tag{9}$$

where

$$\lambda = A_1/A_2. \tag{10}$$

2. *Small vibrations superposed on finite deformations*

The finitely deformed cylinder is now exposed to small vibrations characterized by the displacement field $W_i = W_i(r, \theta, z, t)$, $i = 1, 2, 3$. The equations governing the secondary motions of the tube are the incompressibility condition

$$W_{1,r} + \frac{1}{r}W_1 + \frac{1}{r^2}W_{2,\theta} + W_{3,z} = 0 \tag{11}$$

and the three equations of motion in r, θ and z directions

$$\frac{A_1^2\bar{K}+r^2}{r^2}W_{1,rr} + \left(\alpha^2 + \frac{1}{A_1^2\bar{K}+r^2}\right)W_{1,\theta\theta} + W_{1,zz} + \left[\frac{r^4 - 2A_1^4\bar{K}^2}{r^3(A_1^2\bar{K}+r^2)} - \alpha^2 r\right]W_{1,r} + 2\alpha W_{1,\theta z}$$
$$- \left(\alpha^2 + \frac{1}{A_1^2\bar{K}+r^2}\right)W_1 - \frac{2}{r}\left(\alpha^2 + \frac{1}{A_1^2\bar{K}+r^2}\right)W_{2,\theta} - 2\frac{\alpha}{r}W_{2,z} + \frac{1}{\phi}p'_{,r} = \frac{\rho}{\phi}W_{1,tt}, \tag{12}$$

$$\left[\alpha^2 r + \frac{2r^4 - A_1^4\bar{K}^2}{r^3(A_1^2\bar{K}+r^2)}\right]W_{1,\theta} + 2\alpha r W_{1,z} + \frac{A_1^2\bar{K}+r^2}{r^2}W_{2,rr} + \left(\alpha^2 + \frac{1}{A_1^2\bar{K}+r^2}\right)W_{2,\theta\theta} + W_{2,zz}$$
$$+ 2\alpha W_{2,\theta z} - \frac{3A_1^2\bar{K}+r^2}{r^3}W_{2,r} + 4\frac{A_1^2\bar{K}}{r^4}W_2 + \frac{1}{\phi}p'_{,\theta} = \frac{\rho}{\phi}W_{2,tt}, \tag{13}$$

$$-\left[\alpha^2 r+\frac{A_1^4\bar{K}^2}{r^3(A_1^2\bar{K}+r^2)}\right]W_{1,z}+\frac{A_1^2\bar{K}+r^2}{r^2}W_{3,rr}+\left(\alpha^2+\frac{1}{A_1^2\bar{K}+r^2}\right)w_{3,\theta\theta}+W_{3,zz}$$
$$+2\alpha W_{3,\theta z}+\frac{r^2-A_1^2\bar{K}}{r^3}W_{3,r}+\frac{1}{\phi}p'_{,z}=\frac{\rho}{\phi}W_{3,tt} \qquad (14)$$

where $p'=p'(r,\theta,z,t)$ is an unknown pressure associated with the secondary displacement field and ρ denotes the current mass density.

The associated boundary conditions

$$\frac{A_1^2\bar{K}+r^2}{r^2}W_{1,r}+\frac{1}{2\phi}p'=0,$$
$$W_{1,\theta}+W_{2,r}-\frac{2}{r}W_2=0, \quad \text{on } r=a_1,a_2 \qquad (15)$$
$$W_{1,z}+W_{3,r}=0$$

are obtained from the requirement that the secondary surface tractions vanish.

The solutions to the functions W_i and p' are assumed in the form

$$W_1(r,\theta,z,t)=i\sum_{n=-\infty}^{+\infty}U_{1n}(r)\,e^{i(n\theta+kz+\omega_n t)},$$
$$W_2(r,\theta,z,t)=\sum_{n=-\infty}^{+\infty}U_{2n}(r)\,e^{i(n\theta+kz+\omega_n t)},$$
$$W_3(r,\theta,z,t)=\sum_{n=-\infty}^{+\infty}U_{3n}(r)\,e^{i(n\theta+kz+\omega_n t)}, \qquad (16)$$
$$p'(r,\theta,z,t)=i\sum_{n=-\infty}^{+\infty}U_{4n}(r)\,e^{i(n\theta+kz+\omega_n t)}.$$

Substituting eqns (16) into eqns (11)–(14) and then eliminating $U_{3n}(r)$ from the resulting set of one first order and three second order, linear, ordinary differential equations, a set of three second order equations in U_{1n}, U_{2n} and U_{4n} is obtained:

$$\left[\alpha^2 r+\frac{A_1^4\bar{K}^2+(A_1^2\bar{K}+r^2)^2}{r^3(A_1^2\bar{K}+r^2)}\right]U_1''+\left[\alpha^2\right.$$
$$\left.+\frac{2r^6-2A_1^2\bar{K}(A_1^4\bar{K}^2+7r^2A_1^2\bar{K}+2r^4)-(A_1^2\bar{K}+r^2)^3+\alpha^2r^4(A_1^2\bar{K}+r^2)(3A_1^2\bar{K}+r^2)}{r^4(A_1^2\bar{K}+r^2)^2}\right]U_1'$$
$$+\left[\frac{\rho\omega^2}{\phi r}\left(1-\frac{2A_1^2\bar{K}}{A_1^2\bar{K}+r^2}\right)-\alpha^2k^2r+2A_1^2\bar{K}\frac{\alpha^2+(\alpha n+k)^2}{2(A_1^2\bar{K}+r^2)}-\frac{k^2A_1^4\bar{K}^2}{r^3(A_1^2\bar{K}+r^2)}\right.$$
$$+\frac{2(n^2+1)(A_1^2\bar{K}-r^2)-n^2(A_1^2\bar{K}+r^2)}{r(A_1^2\bar{K}+r^2)^2}$$
$$\left.+\frac{3A_1^2\bar{K}+r^2}{r^5}-\frac{(\alpha n+k)^2}{r}\right]U_1+\frac{n(A_1^2\bar{K}+r^2)}{r^4}U_2''$$
$$+\left[\frac{2\alpha(\alpha n+k)}{r}+\frac{2n}{r(A_1^2\bar{K}+r^2)}-\frac{n(5A_1^2\bar{K}+3r^2)}{r^5}\right]U_2'$$
$$+\left[\frac{n\rho\omega^2}{\phi r^2}+\frac{2n(A_1^2\bar{K}-3r^2)-n^3(A_1^2\bar{K}+r^2)+2\alpha(\alpha n+k)(A_1^4\bar{K}^2-r^4)}{r^2(A_1^2\bar{K}+r^2)^2}\right.$$
$$\left.+\frac{4n(2A_1^2\bar{K}+r^2)-nr^4(\alpha n+k)^2}{r^6}\right]U_2$$
$$-\frac{1}{\phi}U_4''-\frac{2A_1^2\bar{K}}{\phi r(A_1^2\bar{K}+r^2)}U_4'+\frac{k^2}{\phi}U_4=0, \qquad (17)$$

$$\frac{A_1^2\bar{K}+r^2}{r^2}U_1''+\left[\frac{r^4-2A_1^4\bar{K}^2}{r^3(A_1^2\bar{K}+r^2)}-\alpha^2 r\right]U_1'+\left[\frac{\rho\omega^2}{\phi}-k^2-2\alpha nk-\frac{n^2+1}{A_1^2\bar{K}+r^2}-\alpha^2(n^2+1)\right]U_1$$
$$-\left[\frac{2\alpha(\alpha n+k)}{r}+\frac{2n}{r(A_1^2\bar{K}+r^2)}\right]U_2+\frac{1}{\phi}U_4'=0, \quad (18)$$

$$-\left[\alpha^2 nr+2\alpha kr+\frac{n(2r^4-A_1^4\bar{K}^2)}{r^3(A_1^2\bar{K}+r^2)}\right]U_1+\frac{A_1^2\bar{K}+r^2}{r^2}U_2''-\frac{3A_1^2\bar{K}+r^2}{r^3}U_2'$$
$$+\left[\frac{\rho\omega^2}{\phi}-(\alpha n+k)^2+\frac{4A_1^2\bar{K}}{r^4}-\frac{n^2}{A_1^2\bar{K}+r^2}\right]U_2-\frac{n}{\phi}U_4=0 \quad (19)$$

where subscript n has been omitted for convenience.

The corresponding boundary conditions, eqn (15), reduce to

$$\frac{A_1^2\bar{K}+r^2}{r^2}U_1'+\frac{1}{2\phi}U_4=0, \quad (20)$$

$$-nU_1+U_2'-\frac{2}{r}U_2=0, \quad (21)$$

$$\left[\frac{A_1^2\bar{K}(3A_1^2\bar{K}+2r^2)}{r(A_1^2\bar{K}+r^2)^2}+\frac{\alpha^2 r^3}{(A_1^2\bar{K}+r^2)}\right]U_1'+\left[-\frac{\rho\omega^2 r^2}{\phi(A_1^2\bar{K}+r^2)}+\frac{r^2[\alpha^2+(\alpha n+k)^2]}{(A_1^2\bar{K}+r^2)}+\frac{r^2(n^2+1)}{(A_1^2\bar{K}+r^2)}\right.$$
$$\left.+k^2-\frac{1}{r^2}\right]U_1+\frac{n}{r^2}U_2'+\left[\frac{2\alpha(\alpha n+k)r}{(A_1^2\bar{K}+r^2)}+\frac{2nr}{(A_1^2\bar{K}+r^2)^2}-\frac{2n}{r^3}\right]U_2-\frac{r^2}{\phi(A_1^2\bar{K}+r^2)}U_4'=0 \quad (22)$$

on $r=a_1, a_2$.

In the absence of initial twist ($\alpha=0$) and for $k=0$ the above equations compare with those obtained in [10]. When $\bar{K}=0$ the problem becomes identical with the one treated in [11].

The system of eqns (17)–(22) is solved by the method of complementary functions[16]. For this purpose eqns (17)–(19) are first non-dimensionalized by introducing

$$\eta=r/A_1, \quad \bar{\alpha}=\alpha A_1, \quad \bar{k}=kA_1, \quad \bar{\omega}^2=\rho\omega^2 A_1^2/\phi,$$
$$y_4=U_1/A_1, \quad y_5=U_2/A_1^2, \quad y_6=U_4/\phi, \quad (23)$$

and then transformed into a set of six first order equations

$$y_i'(\eta)=F_i(y_j,\eta), \quad i,j=1,2,\ldots,6 \quad (24)$$

where

$$F_1=\left[\frac{\eta^3\bar{\alpha}^2}{\bar{K}+\eta^2}-\frac{\eta^4-2\bar{K}^2}{\eta(\bar{K}+\eta^2)^2}\right]y_1-\frac{\eta^2}{\bar{K}+\eta^2}y_3-\frac{\eta^2}{\bar{K}+\eta^2}\left[\bar{\omega}^2+\bar{k}^2-2n\bar{\alpha}\bar{k}-\frac{n^2+1}{\eta^2+\bar{K}}-(n^2+1)\bar{\alpha}^2\right]y_4$$
$$+\frac{\eta^2}{\bar{K}+\eta^2}\left[\frac{2\bar{\alpha}(n\bar{\alpha}+\bar{k})}{\eta}+\frac{2n}{\eta(\bar{K}+\eta^2)}\right]y_5, \quad (25)$$

$$F_2=\frac{3\bar{K}+\eta^2}{\eta(\bar{K}+\eta^2)}y_2+\frac{\eta^2}{\bar{K}+\eta^2}\left[n\bar{\alpha}^2\eta+2\bar{\alpha}\bar{k}\eta+\frac{n(2\eta^4-\bar{K}^2)}{\eta^3(\bar{K}+\eta^2)}\right]y_4$$
$$-\frac{\eta^2}{\bar{K}+\eta^2}\left[\bar{\omega}^2-(n\bar{\alpha}+\bar{k})^2+\frac{4\bar{K}}{\eta^4}-\frac{n^2}{\bar{K}+\eta^2}\right]y_5+\frac{n\eta^2}{\bar{K}+\eta^2}y_6, \quad (26)$$

$$F_3=\left[\frac{\bar{\alpha}^2\eta^4(\bar{K}+\eta^2)+(\bar{K}+\eta^2)^2+\bar{K}^2}{\eta^3(\bar{K}+\eta^2)}\right]F_1+\frac{n(\eta^2+\bar{K})}{\eta^4}F_2$$
$$+\left[\frac{\eta^6-3\bar{K}^3-17\bar{K}^2\eta^2-7\bar{K}\eta^4+\bar{\alpha}^2\eta^4(3\bar{K}+\eta^2)(\bar{K}+\eta^2)}{\eta^4(\bar{K}+\eta^2)^2}\right]y_1$$

$$+\left[\frac{2\bar{\alpha}(n\bar{\alpha}+\bar{k})}{\eta}+\frac{2n}{\eta(\bar{K}+\eta^2)}-\frac{n(5\bar{K}+3\eta^2)}{\eta^5}\right]y_2-\frac{2\bar{K}}{\eta(\eta^2+\bar{K})}y_3$$
$$+\left[\frac{\eta^2-\bar{K}}{\eta(\bar{K}+\eta^2)}\bar{\omega}^2-\bar{\alpha}^2\bar{k}^2\eta+\frac{2\bar{K}[\bar{\alpha}^2+(\bar{\alpha}n+\bar{k})^2]}{\eta(\eta^2+\bar{K})}-\frac{\bar{k}^2\bar{K}^2}{\eta^3(\bar{K}+\eta^2)}+\frac{3\bar{K}+\eta^2}{\eta^5}-\frac{(\bar{\alpha}n+\bar{k})^2}{\eta}\right.$$
$$\left.+\frac{\bar{K}(n^2+2)-\eta^2(3n^2+2)}{\eta(\eta^2+\bar{K})^2}\right]y_4+\left[\frac{n}{\eta^2}\bar{\omega}^2+\frac{4n(2\bar{K}+\eta^2)-n\eta^4(n\bar{\alpha}+\bar{k})^2}{\eta^6}\right.$$
$$\left.+\frac{2n(\bar{K}-3\eta^2)-n^3(\bar{K}+\eta^2)+2\bar{\alpha}(n\bar{\alpha}+\bar{k})(\bar{K}-\eta^4)}{\eta^2(\bar{K}+\eta^2)^2}\right]y_5+\bar{k}^2y_6, \tag{27}$$

$$F_4=y_1=y_4', \tag{28}$$

$$F_5=y_2=y_5', \tag{29}$$

$$F_6=y_3=y_6'. \tag{30}$$

The boundary conditions, eqns (20)–(22), reduce to

$$\frac{\bar{K}+\eta^2}{\eta^2}y_1+\frac{1}{2}y_6=0, \tag{31}$$

$$y_2-ny_4-\frac{2}{\eta}y_5=0,$$

$$\left[\frac{\bar{K}(3\bar{K}+2\eta^2)}{\eta(\bar{K}+\eta^2)^2}+\frac{\bar{\alpha}^2\eta^3}{\bar{K}+\eta^2}\right]y_1+\frac{n}{\eta^2}y_2-\frac{\eta^2}{\bar{K}+\eta^2}y_3+\left[-\bar{\omega}^2\frac{\eta^2}{\bar{K}+\eta^2}+\frac{\eta^2[\bar{\alpha}^2+(n\bar{\alpha}+\bar{k})^2]}{\bar{K}+\eta^2}\right.$$
$$\left.+\frac{(n^2+1)\eta^2}{(\bar{K}+\eta^2)^2}+\bar{k}^2-\frac{1}{\eta^2}\right]y_4+\left[\frac{2\bar{\alpha}\eta(n\bar{\alpha}+\bar{k})}{\bar{K}+\eta^2}+\frac{2n\eta}{(\bar{K}+\eta^2)^2}-\frac{2n}{\eta^3}\right]y_5=0 \tag{32}$$

on $\eta=a_1/A_1,\ a_2/A_1$.

Equations (24) are integrated numerically by the Runge–Kutta method of order two six times to give six sets of solutions $y_i^{(j)}$, $j=1,2,\ldots,6$, such that the jth set $y_i^{(j)}$ satisfies the initial conditions

$$y_i^{(j)}=\delta_{ij},\qquad i,j=1,2,\ldots,6 \tag{33}$$

on $\eta=\eta_1=a_1/A_1$. δ_{ij} is the Kronecker delta function. The general solution of the system (24) is the linear combination

$$y_i(\eta)=\sum_j B_j y_i^{(j)},\qquad i,j=1,2,\ldots,6. \tag{34}$$

Substituting eqn (34) into the boundary conditions, eqns (31)–(33), a 6×6 characteristic determinant is obtained. For non-trivial solutions, this determinant, which contains the parameters $\bar{\alpha}$, $\bar{k}$, $\bar{K}$, n, A_1/A_2 and $\bar{\omega}^2$, is required to vanish.

ILLUSTRATIVE EXAMPLE AND DISCUSSION OF THE RESULTS

When $n=k=0$ eqns (11)–(14) and the boundary conditions (15) become uncoupled upon substitution of eqn (16):

In radial direction

$$\left[\frac{A_1{}^2\bar{K}(3A_1{}^2\bar{K}+2r^2)}{r^3(A_1{}^2\bar{K}+r^2)}\right]U_1'+\left[-\frac{\rho\omega^2}{\phi}+\frac{1}{A_1{}^2\bar{K}+r^2}-\frac{A_1{}^2\bar{K}+r^2}{r^4}\right]U_1-\frac{1}{\phi}U_4'=0$$

$$U_1'+\frac{1}{r}U_1=0 \tag{35}$$

subjected to

$$\frac{A_1^2\bar{K}}{r^2}U_1'+\frac{1}{2\phi}U_4=0 \quad \text{on} \quad r=a_1, a_2; \tag{36}$$

In transverse direction

$$\frac{A_1^2\bar{K}+r^2}{r^2}U_2''-\frac{3A_1^2\bar{K}+r^2}{r^3}U_2'+\left[\frac{\rho\omega^2}{\phi}+4\frac{A_1^2\bar{K}}{r^4}\right]U_2=0 \tag{37}$$

subjected to

$$U_2'-\left(\frac{2}{r}\right)U_2=0 \quad \text{on} \quad r=a_1, a_2. \tag{38}$$

In axial direction

$$\frac{A_1^2\bar{K}+r^2}{r^2}U_3''+\left(\frac{r^2-A_1^2K}{r^3}\right)U_3'+\frac{\rho\omega^2}{\phi}U_3=0 \tag{39}$$

subjected to

$$U_3'=0 \quad \text{on} \quad r=a_1, a_2. \tag{40}$$

The solution of the system given by eqns (35) and (36) corresponding to breathing motions of the tube is identical with eqn (29) of Ref. [10]. It is seen that the frequencies of pure radial, torsional and longitudinal oscillations are independent of the initial angle of twist α and they are not pursued any further in this study.

For $n \neq 0$, the numerical scheme outlined in the previous section is used since closed form solutions do not seem possible.

Figure 1 shows the non-dimensionalized frequency $\bar{\omega}$ as a function of the initial radial deformation parameter $\bar{K}$ for $A_1/A_2 = 0.80$, $n = -2$. It is seen that $\bar{\omega}$ vs $\bar{K}$ curves depend both on the amount of the initial twist $\bar{\alpha}$ and the axial wave number $\bar{k}$. For a fixed axial wave number and $\bar{k} \neq 0$ the frequencies decrease with increasing initial twist. When $\bar{k} = 0$, the frequencies increase with increasing initial twist.

For the same tube, $\bar{\omega}$ vs $\bar{k}$ curves for $n = +2$ are shown in Fig. 2. In this case, a larger initial twist results in a higher frequency when $\bar{K}$ and $\bar{k}$ are held fixed. It is also observed from Fig. 2 that when $\bar{\alpha} = 0$, the $\bar{\omega}$ vs $\bar{K}$ curves are identical for $n = +2$ and $n = -2$. When $\bar{k} = 0$, $n = +2$

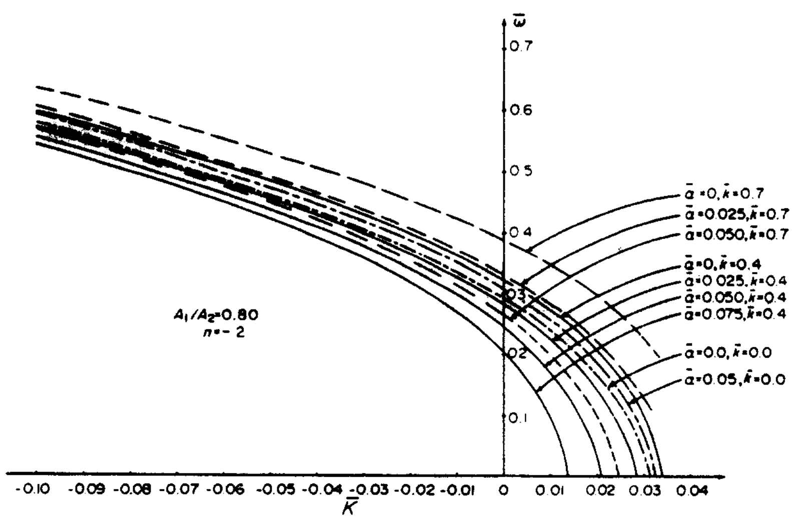

Fig. 1. Frequency vs initial radial deformation parameter for various amounts of initial twist and axial wave numbers.

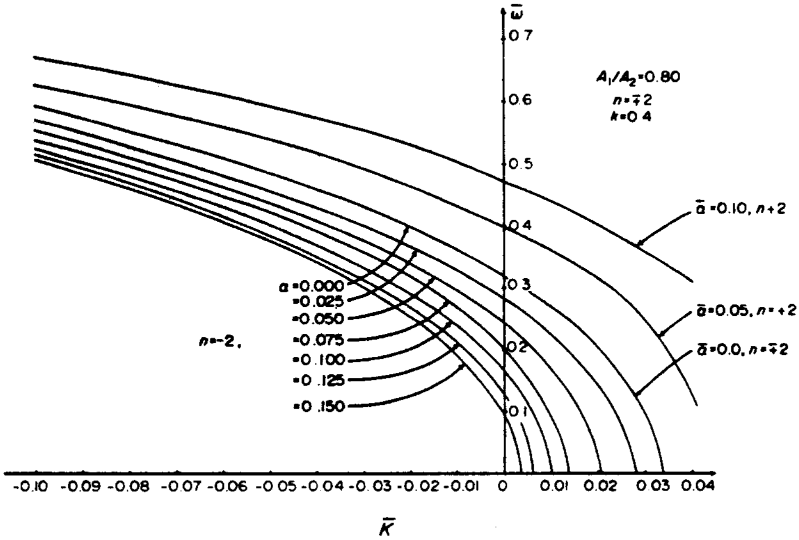

Fig. 2. Frequency vs initial radial deformation parameter for various amounts of initial twist and axial wave numbers.

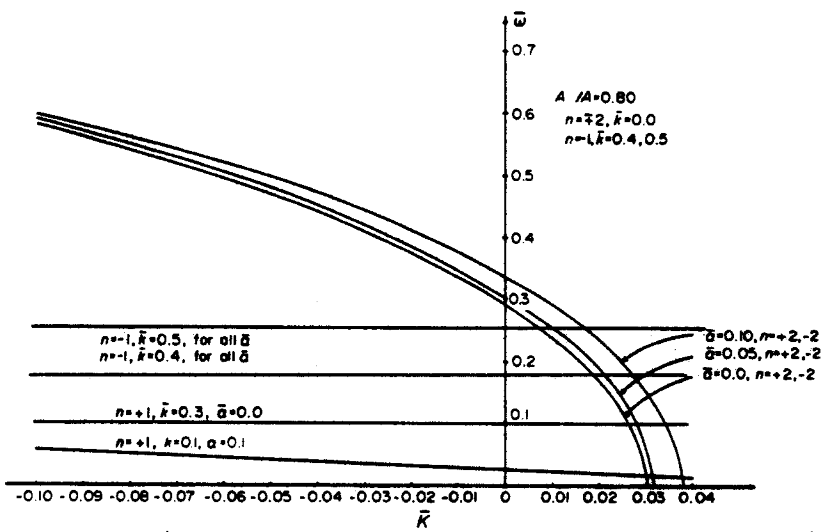

Fig. 3. Frequency vs initial radial deformation parameter for various amounts of initial twist and axial and circumferential wave numbers.

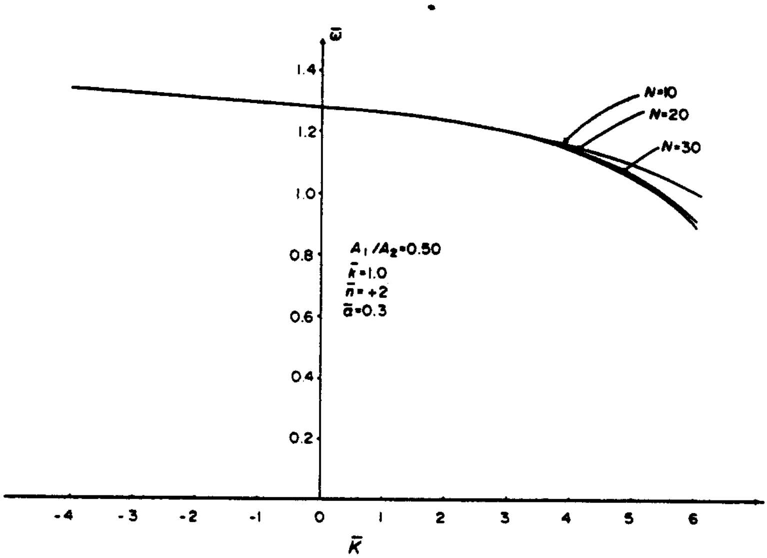

Fig. 4. Frequencies vs initial radial deformation parameter for 10, 20 and 30 pivotal points along the radial coordinate.

and $n = -2$ give identical $\bar{\omega}$ vs $\bar{K}$ curves for all values of $\bar{\alpha}$ (Fig. 3). From Fig. 3, it is seen that the frequencies are independent of the radial deformation parameter $\bar{K}$ for $n = -1$ and for the case $n = +1$, $\bar{\alpha} = 0$, $\bar{k} = 0$. In the figure the plot of the case $n = +1$, $\bar{\alpha} = 0.1$, $\bar{k} = 0.1$ shows that the frequencies decrease with increasing $\bar{K}$. For $|n| > 2$, the $\bar{\omega}$ vs $\bar{K}$ curves show similar patterns to $|n| = 2$ curves although the frequencies are much higher. These curves are not shown here.

Oscillatory motions of the tube exist when $\bar{\omega}$ is real. When $\bar{\omega}$ becomes imaginary, deformations increase without bound. Thus a critical state of prestress is defined for $\bar{\omega}^2 = 0$, i.e. when $\bar{\omega}$ changes from real to imaginary.

For $n = -2$, Fig. 1 shows that the tube becomes unstable at a smaller external pressure when a large initial twist is applied while $\bar{k} \neq 0$ is held fixed. This behavior is completely reversed if the axial wave number is zero and/or $n = +2$.

In obtaining the curves in Figs. 1–3, 20 pivotal points along the radius were used in the numerical integration scheme. Figure 4 shows $\bar{\omega}$ vs $\bar{K}$ curves for a thicker tube obtained by using 10, 20 and 30 pivotal points. A good convergence is achieved at 20 pivotal points.

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